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Some Properties of Quantum Probability*

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Abstract: The following conclusions are established in this paper. The supremum of the absolute value of the absolute variances of an operator in a rank-one projection is equal to the square of the distance from the operator to the scalar operators; a density operator is faithful if and only if it is injective; the strong convergence on the closure of the range of a density operator ρ implies the ρ-a.s. convergence of the sequence; a sequence of operators is strongly convergent if it is ρ-a.s. convergent for every density operator ρ; if ρ is a faithful density operator, then a sequence operators is strongly convergent to A if and only if it is uniformly bounded and convergent to A a.s. $[\rho]$.

Keywords: expectation; variance; density operator; strong convergence; a.s. convergence

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1 Introduction

The classical probability theory was subsumed into the classical measure theory by Kolmogorov in 1933. Quantum theory as nonclassical probability theory was incorporated into the beginnings of noncommutative measure theory by von Neumann in the early thirties. This type of work has also been called noncommutative probability or quantum probability theory. The main applications are in quantum mechanics, statistical mechanics and quantum field theory. Since the framework deals with Hilbert space operators, it is also of interest to operator theorists. A lot of work has been devoted to probability theory on operator algebras, such as C^*-algebras and von Neumann algebras[1-5]. The article [6] gave a survey of various aspects of operator probability theory. A combination of quantum theory and computation as well as probability is a powerful tool in numerical analysis[7].

This note discusses some properties of expectations and variances, gives some necessary and sufficient conditions for a density operator to be faithful, and obtains some relationships between convergence of sequences of operators in operator probability.

2 Some properties of expectations and variances

In what follows, H denotes a separable infinite dimensional complex Hilbert space and $\mathcal{B}(H)$ is the set of bounded linear operators on H. The set of density operators (i.e., positive linear operators of trace 1) on H is denoted by $\mathcal{D}(H)$, and $\mathcal{R}(T)$ is the range of an operator T. The trace of a trace class operator T is denoted by $\mathrm{tr}\,(T)$.

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Definition 2.1 Let $A \in \mathcal{B}(H)$ and $\rho \in \mathcal{D}(H)$. Then the expectation $E_\rho(A)$ of A in the state ρ is defined by $E_\rho(A) = \mathrm{tr}(\rho A)$, the ρ-variance of A is defined by

$$\mathrm{Var}_\rho(A) = E_\rho\big((A - E_\rho(A)I)^2\big),$$

the ρ-absolute variance of A is defined by

$$\big|\mathrm{Var}_\rho\big|(A) = E_\rho\big(\,|A - E_\rho(A)I|^2\,\big).$$

Theorem 2.1 For $A \in \mathcal{B}(H)$, we have

1) $\forall\, x \in H$ with $\|x\| = 1$, $P_x := x \otimes x \in \mathcal{D}(H)$ and

$$E_{P_x}(A) = \langle Ax, x\rangle, \quad |\mathrm{Var}|(A) = \|Ax\|^2 - |\langle Ax, x\rangle|^2;$$

2) There exists a complex number λ_0 such that

$$\sup\{\,|\mathrm{Var}_{P_x}|(A) : x \in H,\ \|x\| = 1\,\} = \inf\{\,\|A - \lambda\|^2 : \lambda \in \mathbb{C}\,\} = \|A - \lambda_0\|^2.$$

Proof 1) We compute that

$$E_{P_x}(A) = E_{P_x}(A) = \mathrm{tr}(P_x A) = \mathrm{tr}\big((x \otimes x)A\big) = \mathrm{tr}(Ax \otimes x) = \langle Ax, x\rangle.$$

Hence

$$\big|\mathrm{Var}_{P_x}\big|(A) = E_{P_x}(\,|A|^2\,) - |E_{P_x}(A)|^2 = \langle\,|A|^2 x, x\rangle - |\langle Ax, x\rangle|^2 = \|Ax\|^2 - |\langle Ax, x\rangle|^2.$$

2) By conclusion 1), we observe that

$$0 \le \big|\mathrm{Var}_{P_x}\big|(A) = \big\|(A-\lambda)x\big\|^2 - \big|\langle (A-\lambda)x, x\rangle\big|^2 \le \|A - \lambda\|^2,$$

where $x \in H$, $\|x\| = 1$ and $\lambda \in \mathbb{C}$. Thus

$$\alpha(A) := \sup\{\,|\mathrm{Var}_{P_x}|(A) : x \in H,\ \|x\| = 1\,\} \le \beta(A) := \inf\{\,\|A - \lambda\|^2 : \lambda \in \mathbb{C}\,\}.$$

Let $W_0(T)$ be the maximal numerical range of an operator $T \in \mathcal{B}(H)$, which is the set of all complex numbers λ such that $\lambda = \lim\limits_{n\to\infty}\langle Tx_n, x_n\rangle$ for a sequence $\{x_n\}$ of unit vectors in H with $\|Tx_n\| \to \|T\|$ as $n \to \infty$. It has been proved that $W_0(T)$ is not empty and there exists a $\lambda_0 \in \mathbb{C}$ such that $0 \in W_0(T - \lambda_0)$ ([8]). Take a $\lambda_0 \in \mathbb{C}$ such that $0 \in W_0(A - \lambda_0)$. Thus, there exists a sequence $\{x_n\}$ of unit vectors in H such that $\lim\limits_{n\to\infty}\langle (A - \lambda_0)x_n, x_n\rangle = 0$ and $\|(A - \lambda_0)x_n\| \to \|A - \lambda_0\|$ as $n \to \infty$. Thus

$$\begin{aligned}\beta(A) \;&\le\; \|A - \lambda_0\|^2 = \lim_{n\to\infty}\big(\,\|(A-\lambda_0)x_n\|^2 - |\langle (A-\lambda_0)x_n, x_n\rangle|^2\,\big)\\ &=\; \lim_{n\to\infty}\big|\mathrm{Var}_{P_{x_n}}\big|(A - \lambda_0) = \lim_{n\to\infty}\big|\mathrm{Var}_{P_{x_n}}\big|(A) \le \alpha(A).\end{aligned}$$

This implies that $\alpha(A) = \beta(A)$. Lastly, the continuity of the function $f(\lambda) = \|A - \lambda\|$ and the fact that $\lim_{|\lambda|\to\infty}\|A - \lambda\| = +\infty$ imply that there exists a $\lambda_0 \in \mathbb{C}$ such that $\|A - \lambda_0\| \le \|A - \lambda\|$ for all $\lambda \in \mathbb{C}$. This completes the proof.

The elements of $\mathcal{D}(H)$ of the form P_x are said to be pure states on H. It follows from the spectral theorem of compact normal operators that every element ρ of $\mathcal{D}(H)$ can be represented as

$$\rho = \sum_{i=1}^{\infty} \lambda_i P_{x_i}, \tag{1}$$

where $\lambda_i \geq 0$, $\sum\limits_{i=1}^{\infty} \lambda_i = 1$ and $\{x_i : i = 1, 2, \cdots\}$ is an orthonormal basis for H.

Definition 2.2 We say that a density operator ρ in $\mathcal{B}(H)$ is faithful if $E_\rho(A) = \mathrm{tr}(\rho A) > 0$ for every nonzero positive operator A in $\mathcal{B}(H)$.

Theorem 2.2 Let ρ be as in (1). Then the following statements are equivalent.

(a) ρ is faithful.

(b) $\lambda_i > 0$ for all i and $\overline{\mathcal{R}(\rho)} = H$.

(c) ρ is injective.

Proof (a) $\Rightarrow$ (b): Let ρ be faithful and assume that $\overline{\mathcal{R}(\rho)} \neq H$. Then $\mathcal{N}(\rho) \neq 0$. Clearly, the operator ρ has the matrix decomposition $\rho = \mathrm{diag}(\rho_1, 0)$ with respect to the orthogonal decomposition $H = \mathcal{N}(\rho)^\perp \oplus \mathcal{N}(\rho)$, where ρ_1 is an operator in $B(\mathcal{N}(\rho)^\perp)$. Put

$$A = \mathrm{diag}\,(0, A_{22}) : \mathcal{N}(\rho)^\perp \oplus \mathcal{N}(\rho) \to \mathcal{N}(\rho)^\perp \oplus \mathcal{N}(\rho),$$

where A_{22} is the identity on $\mathcal{N}(\rho)$. Clearly, $0 \leq A \neq 0$, but $\mathrm{tr}(\rho A) = 0$. This contradicts the fact that ρ is faithful. Hence, $\overline{\mathcal{R}(\rho)} = H$. Assume that there exists $i_0 \in \mathbb{N}$ such that $\lambda_{i_0} = 0$, then $\rho x_{i_0} = 0$ and so $\ker(\rho) \neq \{0\}$. This contradicts the fact that $\overline{\mathcal{R}(\rho)} = H$.

(b) $\Rightarrow$ (c): Let $\overline{\mathcal{R}(\rho)} = H$. Then $\ker(\rho) = \mathcal{R}(\rho)^\perp = \{0\}$. Hence, ρ is injective.

(c) $\Rightarrow$ (a): Let (c) hold. Then $\overline{\mathcal{R}(\rho)} = H$ since ρ is positive. Assume that $A \geq 0$ and $\mathrm{tr}(\rho A) = 0$. Then $\mathrm{tr}(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}}) = \mathrm{tr}(\rho A) = 0$. Since $\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} \geq 0$, we conclude that $\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} = 0$. Thus, $\rho^{\frac{1}{2}} A \rho = 0$. Since ρ has dense range, it follows that $\rho^{\frac{1}{2}} A = 0$. Hence, $\rho A = 0$ and then $A\rho = 0$. By using the fact that $\mathcal{R}(\rho)$ is dense in H again, we see that $A = 0$. This shows that ρ is faithful. This completes the proof.

3 Convergence of sequences of operators

In this section, let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(H)$ and $\rho \in \mathcal{D}(H)$ be as in (1).

Definition 3.1 A sequence $\{A_n\}_{n=1}^{\infty}$ is said to be convergent to A almost surely $[\rho]$, written as $A_n \to A$ a.s. $[\rho]$, if $\lim_{n\to\infty} E_\rho(|A_n - A|^2) = 0$. $\{A_n\}_{n=1}^{\infty}$ is said to be ρ-mean convergent to A if $\lim_{n\to\infty} E_\rho(A_n - A) = 0$. We say that $\{A_n\}_{n=1}^{\infty}$ converges strongly to A on $K \subset H$ if $\lim\limits_{n\to\infty} \|(A_n - A)x\| = 0$ for all $x \in K$. $\{A_n\}_{n=1}^{\infty}$ converges strongly to A if it converges strongly to A on H.

Theorem 3.1 (a) If $\{A_n\}_{n=1}^{\infty}$ converges strongly to A on $\overline{\mathcal{R}(\rho)}$ then $A_n \to A$ a.s. $[\rho]$.

(b) If $A_n \to A$ a.s. $[\rho]$ for all $\rho \in D(H)$, then $\{A_n\}_{n=1}^{\infty}$ converges strongly to A.

Proof (a) Let $\{A_n\}_{n=1}^{\infty}$ converge strongly to A on $\overline{\mathcal{R}(\rho)}$. Then it is uniformly bounded. Since $x_i \in \mathcal{R}(\rho)$ for all i, there exists an $M > 0$ such that $\|A_n x_i\| + \|A\| \leq M$ ($\forall\ i, n \in \mathbb{N}$). Since $\rho \in \mathcal{D}(H)$, we have

$$\rho = \sum_{i=1}^{\infty} \lambda_i P_{x_i},$$

where $\lambda_i \geq 0, \sum\limits_{i=1}^{\infty} \lambda_i = 1$ and $\{x_i\}$ is an orthonormal basis for H. Thus, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \lambda_i < \frac{\varepsilon}{2M^2}.$$

Since $\|(A_n - A)x_i\| \to 0$ as $n \to \infty$ for $i = 1, 2, \cdots, N$, there exists positive integer K such that

$$\sum_{i=1}^{N} \lambda_i \|(A_n - A)x_i\|^2 < \frac{\varepsilon}{2}, \quad \forall\, n > K.$$

Thus, whenever $n > K$, we have

$$\begin{aligned} E_\rho(|A_n - A|^2) &= \operatorname{tr}(\rho|A_n - A|^2) = \sum_{i=1}^{\infty} \lambda_i \|(A_n - A)x_i\|^2 \\ &= \sum_{i=1}^{N} \lambda_i \|(A_n - A)x_i\|^2 + \sum_{i=N+1}^{\infty} \lambda_i \|(A_n - A)x_i\|^2 \\ &\leq \frac{\varepsilon}{2} + M^2 \sum_{i=N+1}^{\infty} \lambda_i \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $A_n \to A$ a.s. $[\rho]$.

(b) For every $x \in H$ with $\|x\| = 1$, $P_x := x \otimes x \in \mathcal{D}(H)$, then $A_n \to A$ a.s. $[P_x]$. By Theorem 2.1, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} E_{P_x}(|A_n - A|^2) = \lim_{n\to\infty} \operatorname{tr}(P_x|A_n - A|^2) \\ &= \lim_{n\to\infty} \langle |A_n - A|^2 x, x\rangle = \lim_{n\to\infty} \langle (A_n - A)x, (A_n - A)x\rangle \\ &= \lim_{n\to\infty} \|(A_n - A)x\|^2. \end{aligned}$$

Thus, $\{A_n\}_{n=1}^{\infty}$ converges strongly to A. This completes the proof.

Corollary 3.1 If $\{A_n\}$ converges strongly to A on $\overline{\mathcal{R}(\rho)}$, then $\{A_n\}$ is ρ-mean convergent to A.

Theorem 3.2 Let ρ be as in (1) with $\lambda_i > 0$ for all i. Then

(a) $A_n \to A$ strongly on $\overline{\mathcal{R}(\rho)}$ if and only if $\{A_n\}$ is uniformly bounded and $A_n \to A$ a.s. $[\rho]$.

(b) In the case where ρ is faithful, $A_n \to A$ strongly if and only if $\{A_n\}$ is uniformly bounded and $A_n \to A$ a.s. $[\rho]$.

Proof (a) By Theorem 3.1, it suffices to prove the sufficiency. Assume that $A_n \to A$ a.s. $[\rho]$ and there exists an $M > 0$ such that $\|A_n\| \leq M$ for all n. Since

$$\begin{aligned} E_\rho(|A_n - A|^2) &= \operatorname{tr}(\rho(|A_n - A|^2)) = \sum_{i=1}^{\infty} \langle \rho(|A_n - A|^2)x_i, x_i\rangle \\ &= \sum_{i=1}^{\infty} \lambda_i \langle (|A_n - A|^2)x_i, x_i\rangle = \sum_{i=1}^{\infty} \lambda_i \|(A_n - A)x_i\|^2 \\ &\geq \lambda_i \|(A_n - A)x_i\|^2, \end{aligned}$$

and $\lambda_i > 0$ for all i, we conclude that $A_n x_i \to Ax_i$ as $n \to \infty$ for every i and so $A_n x \to Ax$ as $n \to \infty$ for every $x \in \mathcal{R}(\rho)$. For any $x \in \overline{\mathcal{R}(\rho)}$ and every $\varepsilon > 0$, there exists a $y \in \mathcal{R}(\rho)$ such that $\|x - y\| < \varepsilon$. Take an $N \in \mathbb{N}$ such that $\|A_n y - Ay\| < \varepsilon$ for all $n > N$. Thus, whenever $n > N$, we have

$$\begin{aligned} \|A_n x - Ax\| &= \|A_n x - A_n y + A_n y - Ay + Ay - Ax\| \\ &\le \|A_n x - A_n y\| + \|A_n y - Ay\| + \|Ay - Ax\| \\ &\le \|A_n\| \|x - y\| + \|A_n y - Ay\| + \|Ay - Ax\| \\ &\le (M + \|A\| + 1)\varepsilon. \end{aligned}$$

Hence, $A_n \to A$ strongly on $\overline{\mathcal{R}(\rho)}$.

(b) Use (a) and the fact that $\overline{\mathcal{R}(\rho)} = H$.

This completes the proof.

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关于量子概率论的若干研究

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摘 要: 本文证明了以下结论：算子关于一秩投影的绝对方差的绝对值的上确界等于这个算子到所有数乘算子的距离的平方；一个密度算子是忠实的当且仅当它是单射；算子列在密度算子 ρ 的值域的闭包上的强收敛性蕴含它关于 ρ 的 a.s. 收敛性；如果算子列关于每个密度算子是 a.s. 收敛的，那么它一定是强收敛的；算子列 $\{A_n\}$ 强收敛于 A 当且仅当它是一致有界的且关于某个忠实的密度算子 ρ a.s. 收敛于 A。

关键词: 期望；方差；密度算子；强收敛性；a.s. 收敛性